

CONSTRAINED GAUSS VARIATIONAL PROBLEM FOR CONDENSERS WITH TOUCHING PLATES

NATALIA ZORII

ABSTRACT. We study a constrained minimum energy problem with an external field relative to the α -Riesz kernel $|x - y|^{\alpha - n}$ of an arbitrary order $\alpha \in (0, n)$ for a generalized condenser $\mathbf{A} = (A_1, A_2)$ with touching oppositely-charged plates in \mathbb{R}^n , $n \geq 2$. Conditions sufficient for the solvability of the problem are obtained. Our arguments are mainly based on the definition of an appropriate metric structure on a set of vector measures associated with \mathbf{A} and the establishment of a completeness theorem for the corresponding metric space.

1. INTRODUCTION

This paper is devoted to the well-known Gauss variational problem of minimizing the α -Riesz energy, $\alpha \in (0, n)$, in the presence of an external field, treated for a generalized condenser \mathbf{A} with touching oppositely-charged plates $A_1, A_2 \subset \mathbb{R}^n$, $n \geq 2$. In the case where the Euclidean distance $\text{dist}(A_1, A_2)$ between A_1 and A_2 is nonzero (which might happen if A_1 and A_2 touch each other *only* at the Alexandroff point $\omega_{\mathbb{R}^n}$), a fairly complete investigation of this problem has been provided in [17, 18] (see also the bibliography therein; see Section 3.3 below for a short review).

However, the results obtained in [17, 18] and the approach developed are no longer valid if $\text{dist}(A_1, A_2) = 0$ (e.g, if A_1 and A_2 touch each other at a *finite* point $x \in \mathbb{R}^n$). Then the infimum of the Gauss functional can not, in general, be attained among the admissible measures. Using the electrostatic interpretation, which is possible for the Coulomb kernel $|x - y|^{-1}$ on \mathbb{R}^3 , a short-circuit between A_1 and A_2 might occur. Therefore, it is meaningful to ask what kind of additional requirements on the charges (measures) under consideration would prevent this phenomenon.

A natural idea, to be exploited below, is to impose an upper constraint on vector measures associated with \mathbf{A} so that the infimum of the Gauss functional over the corresponding (narrower) class of constrained admissible vector measures would be already an actual minimum. See Section 3.4 for a precise formulation of the constrained problem; as for the history of the question, cf. Remarks 3.10–3.12.

The author expresses her gratitude to Erwin Schrödinger International Institute for providing conducive research atmosphere during her stay when part of this manuscript was written.

A statement on the solvability of the constrained Gauss variational problem is given by Theorem 4.1, the main result of the study. Its proof is based on the definition of an appropriate metric structure on a set of vector measures associated with \mathbf{A} and the establishment of a completeness theorem for the corresponding metric space (see Theorem 5.1). The results obtained are illustrated by Example 4.2.

2. PRELIMINARIES

Let X be a locally compact Hausdorff space, to be specified below, and $\mathfrak{M}(X)$ the linear space of all real-valued scalar Radon measures μ on X , equipped with the *vague* topology, i.e. the topology of pointwise convergence on the class $C_0(X)$ of all real-valued continuous functions on X with compact support. We denote by μ^+ and μ^- the positive and the negative parts in the Hahn–Jordan decomposition of a measure $\mu \in \mathfrak{M}(X)$, respectively, and by S_X^μ its support. These and other notions of the theory of measures and integration in a locally compact space, to be used throughout the paper, can be found in [3, 8] (see also [9] for a short review).

A *kernel* $\kappa(x, y)$ on X is a symmetric, lower semicontinuous function $\kappa : X \times X \rightarrow [0, \infty]$. Given $\mu, \mu_1 \in \mathfrak{M}(X)$, let $E_\kappa(\mu, \mu_1)$ and $U_\kappa^\mu(\cdot)$ denote the *mutual energy* and the *potential* relative to the kernel κ , respectively, i.e.

$$E_\kappa(\mu, \mu_1) := \int \kappa(x, y) d(\mu \otimes \mu_1)(x, y),$$

$$U_\kappa^\mu(x) := \int \kappa(x, y) d\mu(y), \quad x \in X.$$

(When introducing notation, we assume the corresponding object on the right to be well defined — as a finite number or $\pm\infty$.)

For $\mu = \mu_1$, the mutual energy $E_\kappa(\mu, \mu_1)$ defines the *energy* $E_\kappa(\mu) := E_\kappa(\mu, \mu)$. Let $\mathcal{E}_\kappa(X)$ consist of all $\mu \in \mathfrak{M}(X)$ whose energy $E_\kappa(\mu)$ is finite.

Having denoted by $\mathfrak{M}^+(X)$ the convex cone of all nonnegative $\mu \in \mathfrak{M}(X)$, we write $\mathcal{E}_\kappa^+(X) := \mathfrak{M}^+(X) \cap \mathcal{E}_\kappa(X)$. Given a set $B \subset X$, $B \neq X$, let $\mathfrak{M}^+(B; X)$ consist of all $\mu \in \mathfrak{M}^+(X)$ concentrated in B , and let $\mathcal{E}_\kappa^+(B; X) := \mathcal{E}_\kappa(X) \cap \mathfrak{M}^+(B; X)$.

Observe that, if B is closed, then $\mu \in \mathfrak{M}^+(X)$ belongs to $\mathfrak{M}^+(B; X)$ if and only if the set $X \setminus B$ is μ -negligible (or, equivalently, if $S_X^\mu \subset B$). Furthermore, then $\mathfrak{M}^+(B; X)$ and $\mathcal{E}_\kappa^+(B; X)$ are closed in the induced vague topology (see, e.g., [9]).

Let $C_\kappa(B)$ be the *interior capacity* of B relative to the kernel κ , given by

$$C_\kappa(B) := \left[\inf_{\mu \in \mathcal{E}_\kappa^+(B; X): \mu(B)=1} E_\kappa(\mu) \right]^{-1};$$

see, e.g., [9, 13]. Then $0 \leq C_\kappa(B) \leq \infty$. (Here, as usual, the infimum over the empty set is taken to be $+\infty$. We also put $1/(+\infty) = 0$ and $1/0 = +\infty$.)

A kernel κ is called *strictly positive definite* if the energy $E_\kappa(\mu)$, $\mu \in \mathfrak{M}(X)$, is nonnegative whenever defined and $E_\kappa(\mu) = 0$ implies $\mu = 0$. Then $\mathcal{E}_\kappa(X)$ forms a pre-Hilbert space with the scalar product $E_\kappa(\mu, \mu_1)$ and the norm $\|\mu\|_\kappa := \sqrt{E_\kappa(\mu)}$ (see [9]). The topology on $\mathcal{E}_\kappa(X)$ defined by $\|\cdot\|_\kappa$ is said to be *strong*.

Following Fuglede [9], we call a strictly positive definite kernel κ *perfect* if any strong Cauchy sequence in $\mathcal{E}_\kappa^+(X)$ converges strongly and, in addition, the strong topology on $\mathcal{E}_\kappa^+(X)$ is finer than the induced vague topology on $\mathcal{E}_\kappa^+(X)$. Note that then $\mathcal{E}_\kappa^+(X)$ is a strongly complete metric space.

3. UNCONSTRAINED AND CONSTRAINED GAUSS VARIATIONAL PROBLEMS

Throughout the paper, let $n \geq 2$, $n \in \mathbb{N}$, and $\alpha \in (0, n)$ be fixed. In $X = \mathbb{R}^n$, consider the α -Riesz kernel $\kappa_\alpha(x, y) := |x - y|^{\alpha-n}$ of order α , where $|x - y|$ denotes the Euclidean distance between x and y in \mathbb{R}^n . The α -Riesz kernel is known to be strictly positive definite and, moreover, perfect (see [5, 6]); hence, the metric space $\mathcal{E}_{\kappa_\alpha}^+(\mathbb{R}^n)$ is complete in the induced strong topology. However, by Cartan [4] (see also [12, Theorem 1.19]), the whole pre-Hilbert space $\mathcal{E}_{\kappa_\alpha}(\mathbb{R}^n)$ for $\alpha \in (1, n)$ is strongly incomplete (compare with Theorem 5.1 and Remark 5.2 below).

From now on we shall write simply α instead of κ_α if it serves as an index. E.g., $C_\alpha(\cdot) = C_{\kappa_\alpha}(\cdot)$ denotes the α -Riesz interior capacity of a set. An expression $\mathcal{U}(x)$, involving a variable point $x \in \mathbb{R}^n$, is said to subsist *nearly everywhere* (n.e.) in a set $B \subset \mathbb{R}^n$ if $C_\alpha(N) = 0$, where N consists of all $x \in B$ for which $\mathcal{U}(x)$ fails to hold.

3.1. Generalized condensers. Vector measures and their α -Riesz energies. Given $B \subset \mathbb{R}^n$, write $B^c := \mathbb{R}^n \setminus B$. Recall that a (*standard*) *condenser* in \mathbb{R}^n is usually meant as an ordered pair of nonempty, closed (though not necessarily compact), nonintersecting sets in \mathbb{R}^n . We extend this notion as follows.

Definition 3.1. An ordered pair $\mathbf{A} := (A_1, A_2)$ of nonempty sets in \mathbb{R}^n is called a *generalized condenser* if the following two conditions are fulfilled for every $i = 1, 2$:

- (a) $A_i \subset D_i$, where $D_i := (\text{Cl}_{\mathbb{R}^n} A_j)^c$, $j \neq i$;
- (b) A_i is closed in the relative topology of the (open) set D_i .

Observe that the notion of a generalized condenser $\mathbf{A} = (A_1, A_2)$ is reduced to that of a standard one if and only if the sets A_i , $i = 1, 2$, are closed in \mathbb{R}^n .

In the example below, $n = 3$ and $\overline{B}(x, 1)$ is the closed three-dimensional ball of radius 1 centered at $x \in \mathbb{R}^3$.

Example 3.2. Consider $\overline{B}(\xi_1, 1)$ and $\overline{B}(\xi_2, 1)$ with $\xi_1 = (0, 0, 0)$ and $\xi_2 = (2, 0, 0)$; these balls intersect each other at $\xi_0 = (1, 0, 0)$. Then the sets $A_i := \overline{B}(\xi_i, 1) \setminus \{\xi_0\}$, $i = 1, 2$,

satisfy both assumptions (a) and (b) from Definition 3.1 and, hence, form a generalized condenser \mathbf{A} in \mathbb{R}^3 , which certainly is not a standard one.

In all that follows, fix a generalized condenser $\mathbf{A} = (A_1, A_2)$ such that $A_i \neq D_i$ for all $i = 1, 2$. To avoid triviality, suppose $\prod_{i=1,2} C_\alpha(A_i) > 0$.

Let $\mathfrak{M}^+(\mathbf{A})$ stand for the Cartesian product $\prod_{i=1,2} \mathfrak{M}^+(A_i; D_i)$, where D_i is thought of as a locally compact space. Then $\nu \in \mathfrak{M}^+(\mathbf{A})$ is a nonnegative *vector measure* $(\nu^i)_{i=1,2}$ with the components $\nu^i \in \mathfrak{M}^+(A_i; D_i)$; it is said to be *associated* with the condenser \mathbf{A} .

Definition 3.3. The \mathbf{A} -vague topology on $\mathfrak{M}^+(\mathbf{A})$ is the topology of the product space $\prod_{i=1,2} \mathfrak{M}^+(A_i; D_i)$, where each of the factors $\mathfrak{M}^+(A_i; D_i)$, $i = 1, 2$, is endowed with the vague topology induced from $\mathfrak{M}(D_i)$.

As A_i is closed in D_i , $\mathfrak{M}^+(\mathbf{A})$ is \mathbf{A} -vaguely closed. Besides, since every $\mathfrak{M}(D_i)$ is Hausdorff, so is $\mathfrak{M}^+(\mathbf{A})$ (see [11, Chapter 3, Theorem 5]). Hence, an \mathbf{A} -vague limit of any $\{\nu_k\}_{k \in \mathbb{N}} \subset \mathfrak{M}^+(\mathbf{A})$ belongs to $\mathfrak{M}^+(\mathbf{A})$ and is unique (provided it exists).

If $\nu \in \mathfrak{M}^+(\mathbf{A})$ and a vector-valued function $\mathbf{u} = (u_i)_{i=1,2}$ with the ν^i -measurable components $u_i : A_i \rightarrow [-\infty, \infty]$ are given, then we write $\langle \mathbf{u}, \nu \rangle := \sum_{i=1,2} \int u_i d\nu^i$.

We call A_1 and A_2 the *positive* and the *negative plates* of \mathbf{A} , respectively. In accordance with the electrostatic interpretation of a condenser, assume that the interaction between the charges lying on the conductors A_i , $i = 1, 2$, is characterized by the matrix $(s_i s_j)_{i,j=1,2}$, where

$$s_i := \text{sign } A_i = \begin{cases} +1 & \text{if } i = 1, \\ -1 & \text{if } i = 2. \end{cases}$$

Then the α -Riesz mutual energy of $\nu, \nu_1 \in \mathfrak{M}^+(\mathbf{A})$ is given formally by

$$(3.1) \quad E_\alpha(\nu, \nu_1) := \sum_{i,j=1,2} s_i s_j \int |x - y|^{\alpha-n} d(\nu^i \otimes \nu_1^j)(x, y).$$

For $\nu = \nu_1$, $E_\alpha(\nu, \nu_1)$ defines the α -Riesz energy $E_\alpha(\nu) := E_\alpha(\nu, \nu)$ of ν . We denote by $\mathcal{E}_\alpha^+(\mathbf{A})$ the set of all $\nu \in \mathfrak{M}^+(\mathbf{A})$ whose energy $E_\alpha(\nu)$ is finite.

3.2. Metric structure on classes of vector measures. Let $\check{\mathfrak{M}}^+(\mathbf{A})$ consist of all $\nu \in \mathfrak{M}^+(\mathbf{A})$ such that each of its components ν^i , $i = 1, 2$, can be extended to a Radon measure on \mathbb{R}^n (denote it again by ν^i) by setting

$$\nu^i(\varphi) := \langle \chi_{D_i} \varphi, \nu^i \rangle \quad \text{for all } \varphi \in C_0(\mathbb{R}^n),$$

where χ_{D_i} is the characteristic function of D_i . A sufficient condition for $\nu \in \mathfrak{M}^+(\mathbf{A})$ to belong to $\check{\mathfrak{M}}^+(\mathbf{A})$ is that $\nu^i(A_i) < \infty$ for all $i = 1, 2$. Also note that

$$(3.2) \quad \check{\mathfrak{M}}^+(\mathbf{A}) = \mathfrak{M}^+(\mathbf{A}) \iff \mathbf{A} \text{ is standard};$$

otherwise, $\check{\mathfrak{M}}^+(\mathbf{A})$ forms a proper subset of $\mathfrak{M}^+(\mathbf{A})$ that is not \mathbf{A} -vaguely closed.

For any $\boldsymbol{\nu} \in \check{\mathfrak{M}}^+(\mathbf{A})$, write

$$(3.3) \quad R\boldsymbol{\nu} := \sum_{i=1,2} s_i \nu^i;$$

then $R\boldsymbol{\nu}$ is a *signed* scalar Radon measure on \mathbb{R}^n . Since $A_1 \cap A_2 = \emptyset$, R is a one-to-one mapping between $\check{\mathfrak{M}}^+(\mathbf{A})$ and its R -image,

$$R(\check{\mathfrak{M}}^+(\mathbf{A})) = \{\nu \in \mathfrak{M}(\mathbb{R}^n) : \nu^+ \in \mathfrak{M}^+(A_1; D_1), \nu^- \in \mathfrak{M}^+(A_2; D_2)\}.$$

Lemma 3.4. *For any $\boldsymbol{\nu}, \boldsymbol{\nu}_1 \in \check{\mathfrak{M}}^+(\mathbf{A})$, $E_\alpha(\boldsymbol{\nu}, \boldsymbol{\nu}_1)$ is well defined if and only if so is $E_\alpha(R\boldsymbol{\nu}, R\boldsymbol{\nu}_1)$, and then they coincide:*

$$(3.4) \quad E_\alpha(\boldsymbol{\nu}, \boldsymbol{\nu}_1) = E_\alpha(R\boldsymbol{\nu}, R\boldsymbol{\nu}_1).$$

Proof. Indeed, this can be obtained directly from (3.1) and (3.3). \square

In view of the strict positive definiteness of the α -Riesz kernel, Lemma 3.4 yields that $E_\alpha(\boldsymbol{\nu})$, $\boldsymbol{\nu} \in \check{\mathfrak{M}}^+(\mathbf{A})$, is ≥ 0 whenever defined, and it is zero only for $\boldsymbol{\nu} = \mathbf{0}$. Write $\check{\mathcal{E}}_\alpha^+(\mathbf{A}) := \mathcal{E}_\alpha^+(\mathbf{A}) \cap \check{\mathfrak{M}}^+(\mathbf{A})$. Having defined

$$\|\boldsymbol{\nu} - \boldsymbol{\nu}_1\|_{\check{\mathcal{E}}_\alpha^+(\mathbf{A})} := \left[\sum_{i,j=1,2} s_i s_j E_\alpha(\nu^i - \nu_1^i, \nu^j - \nu_1^j) \right]^{1/2} \quad \text{for all } \boldsymbol{\nu}, \boldsymbol{\nu}_1 \in \check{\mathcal{E}}_\alpha^+(\mathbf{A}),$$

we also see from (3.4) by means of a straightforward calculation that, in fact,

$$(3.5) \quad \|\boldsymbol{\nu} - \boldsymbol{\nu}_1\|_{\check{\mathcal{E}}_\alpha^+(\mathbf{A})} = \|R\boldsymbol{\nu} - R\boldsymbol{\nu}_1\|_\alpha,$$

so that $\check{\mathcal{E}}_\alpha^+(\mathbf{A})$ forms a metric space with the metric $\|\boldsymbol{\nu} - \boldsymbol{\nu}_1\|_{\check{\mathcal{E}}_\alpha^+(\mathbf{A})}$. Since, in consequence of (3.5), $\check{\mathcal{E}}_\alpha^+(\mathbf{A})$ and its R -image are isometric, similar to the terminology in $\mathcal{E}_\alpha(\mathbb{R}^n)$ we shall call the topology of the metric space $\check{\mathcal{E}}_\alpha^+(\mathbf{A})$ *strong*.

3.3. Unconstrained \mathbf{f} -weighted minimum α -Riesz energy problem. Given a locally compact space X , let $\Phi(X)$ consist of all lower semicontinuous functions $\psi : X \rightarrow (-\infty, \infty]$ such that $\psi \geq 0$ unless X is compact. Then for any $\psi \in \Phi(X)$, the map

$$\mu \mapsto \langle \psi, \mu \rangle, \quad \mu \in \mathfrak{M}^+(X),$$

is vaguely lower semicontinuous (see, e.g., [9, Section 1.1]).

Fix a vector-valued function $\mathbf{f} = (f_i)_{i=1,2}$, where each $f_i : A_i \rightarrow [-\infty, \infty]$ is universally measurable and it is treated as an *external field* acting on the charges from $\mathfrak{M}^+(A_i; D_i)$. Then the \mathbf{f} -weighted α -Riesz energy of $\boldsymbol{\nu} \in \mathcal{E}_\alpha^+(\mathbf{A})$ is defined by

$$(3.6) \quad G_{\alpha, \mathbf{f}}(\boldsymbol{\nu}) := E_\alpha(\boldsymbol{\nu}) + 2\langle \mathbf{f}, \boldsymbol{\nu} \rangle;$$

$G_{\alpha, \mathbf{f}}(\cdot)$ is also known as the *Gauss functional* (see, e.g., [13]). Let $\mathcal{E}_{\alpha, \mathbf{f}}^+(\mathbf{A})$ consist of all $\boldsymbol{\nu} \in \mathcal{E}_\alpha^+(\mathbf{A})$ with finite $G_{\alpha, \mathbf{f}}(\boldsymbol{\nu})$.

In this paper, we tacitly assume that one of the following Cases I or II holds:

- I. For every $i = 1, 2$, $f_i \in \Phi(A_i)$, where A_i is thought of as a locally compact space;
- II. For every $i = 1, 2$, $f_i = s_i U_\alpha^\zeta|_{A_i}$, where a (signed) scalar measure $\zeta \in \mathcal{E}_\alpha(\mathbb{R}^n)$ is given.

For any $\nu \in \check{\mathcal{E}}_\alpha^+(\mathbf{A})$, $G_{\alpha, \mathbf{f}}(\nu)$ is then well defined in both Cases I and II. Furthermore, if Case II takes place, then, by (3.6) and (3.4),

$$\begin{aligned}
 (3.7) \quad G_{\alpha, \mathbf{f}}(\nu) &= \|R\nu\|_\alpha^2 + 2 \sum_{i=1,2} s_i E_\alpha(\zeta, \nu^i) \\
 &= \|R\nu\|_\alpha^2 + 2E_\alpha(\zeta, R\nu) = \|R\nu + \zeta\|_\alpha^2 - \|\zeta\|_\alpha^2
 \end{aligned}$$

and, consequently,

$$(3.8) \quad -\infty < -\|\zeta\|_\alpha^2 \leq G_{\alpha, \mathbf{f}}(\nu) < \infty \quad \text{for all } \nu \in \check{\mathcal{E}}_\alpha^+(\mathbf{A}).$$

Also fix a numerical vector $\mathbf{a} = (a_i)_{i=1,2}$ with $a_i > 0$ and a vector-valued function $\mathbf{g} = (g_i)_{i=1,2}$, where all the $g_i : D_i \rightarrow (0, \infty)$ are continuous and such that

$$(3.9) \quad g_{i, \inf} := \inf_{x \in A_i} g_i(x) > 0.$$

Write

$$\begin{aligned}
 \mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}) &:= \{\nu \in \mathfrak{M}^+(\mathbf{A}) : \langle g_i, \nu^i \rangle = a_i \text{ for all } i = 1, 2\}, \\
 \mathcal{E}_{\alpha, \mathbf{f}}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}) &:= \mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}) \cap \mathcal{E}_{\alpha, \mathbf{f}}^+(\mathbf{A}), \\
 G_{\alpha, \mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) &:= \inf_{\nu \in \mathcal{E}_{\alpha, \mathbf{f}}^+(\mathbf{A}, \mathbf{a}, \mathbf{g})} G_{\alpha, \mathbf{f}}(\nu).
 \end{aligned}$$

Observe that, because of (3.9),

$$\nu^i(A_i) \leq a_i g_{i, \inf}^{-1} < \infty \quad \text{for all } \nu \in \mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g})$$

and, therefore,

$$(3.10) \quad \mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}) \subset \check{\mathfrak{M}}^+(\mathbf{A}), \quad \mathcal{E}_{\alpha, \mathbf{f}}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}) \subset \check{\mathcal{E}}_\alpha^+(\mathbf{A}).$$

Combined these with Lemma 3.4 and the fact that a lower semicontinuous function is bounded from below on a compact set, in Case I we obtain

$$G_{\alpha, \mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) > -\infty.$$

The same holds true in Case II as well, which is obvious from (3.8) and (3.10).

If the class $\mathcal{E}_{\alpha, \mathbf{f}}^+(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is nonempty or, equivalently, if

$$(3.11) \quad G_{\alpha, \mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) < \infty,$$

then the following (unconstrained) \mathbf{f} -weighted minimum α -Riesz energy problem, also known as the *Gauss variational problem* (see [10, 13]), makes sense.

Problem 3.5. Does there exist $\lambda_{\mathbf{A}} \in \mathcal{E}_{\alpha, \mathbf{f}}^+(\mathbf{A}, \mathbf{a}, \mathbf{g})$ with $G_{\alpha, \mathbf{f}}(\lambda_{\mathbf{A}}) = G_{\alpha, \mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$?

Remark 3.6. Analysis similar to that for a standard condenser (cf. Lemma 6.2 in [17]) shows that assumption (3.11) is equivalent to the following one:

$$f_i(x) < \infty \quad \text{n.e. in } A_i, \quad i = 1, 2.$$

In turn, this yields that (3.11) holds automatically whenever Case II takes place, for the α -Riesz potential of $\zeta \in \mathcal{E}_\alpha(\mathbb{R}^n)$ is finite n.e. in \mathbb{R}^n .

Remark 3.7. In the case where every A_i is compact in D_i (i.e., \mathbf{A} is a compact standard condenser) and Case I takes place, the solvability of Problem 3.5 can easily be established by exploiting the \mathbf{A} -vague topology only, since then $\mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is \mathbf{A} -vaguely compact, while $G_{\alpha, \mathbf{f}}(\cdot)$ is \mathbf{A} -vaguely lower semicontinuous on $\mathcal{E}_{\alpha, \mathbf{f}}^+(\mathbf{A})$ (see [13, Theorem 2.30]). However, these arguments break down if any of the two requirements is not satisfied, and then Problem 3.5 becomes rather nontrivial. E.g., $\mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is no longer \mathbf{A} -vaguely compact if some of the A_i is noncompact in D_i .

Remark 3.8. Assume that \mathbf{A} is still a standard condenser, though now, in contrast to Remark 3.7, its plates might be noncompact in \mathbb{R}^n . Under the assumption

$$(3.12) \quad \text{dist}(A_1, A_2) := \inf_{x \in A_1, y \in A_2} |x - y| > 0,$$

in [17, 18] we worked out an approach based on both the \mathbf{A} -vague and the strong topologies on $\mathcal{E}_\alpha^+(\mathbf{A})$ and a certain strong completeness result, which made it possible to provide a fairly complete analysis of Problem 3.5. In more detail, it has been shown that, if $g_i|_{A_i}$, $i = 1, 2$, are bounded from above, then, in both Cases I and II,

$$(3.13) \quad C_\alpha(A_1 \cup A_2) < \infty$$

is sufficient for Problem 3.5 to be (uniquely) solvable for every \mathbf{a} (see [17, Theorem 8.1]). However, if (3.13) does not hold, then, in general, there exists a vector \mathbf{a}' such that the Gauss variational problem admits no solution [17]. Therefore, it was interesting to give a description of the set of all vectors \mathbf{a} for which the problem would be, nevertheless, solvable. Such a characterization has been established in [18].

In the rest of the paper, except for Remark 3.10, we do not assume (3.12) necessarily to hold. Then the results obtained in [17, 18] and the approach developed are no longer valid. In particular, assumption (3.13) does not guarantee anymore that $G_{\alpha, \mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is attained among $\nu \in \mathcal{E}_{\alpha, \mathbf{f}}^+(\mathbf{A}, \mathbf{a}, \mathbf{g})$. Using the electrostatic interpretation, a short-circuit between the touching oppositely-charged plates of the condenser might occur. Therefore, it is meaningful to ask what kind of additional requirements on the measures under consideration would prevent this phenomenon, and a solution to the corresponding \mathbf{f} -weighted minimum α -Riesz energy problem would, nevertheless, exist.

The idea discussed below is to find out such an upper constraint on the measures from $\mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g})$ which would not allow the "blow-up" effect between A_1 and A_2 .

3.4. Constrained \mathbf{f} -weighted minimum α -Riesz energy problem. Let $\mathfrak{C}(\mathbf{A})$ consist of all $\sigma = (\sigma^i)_{i=1,2} \in \mathfrak{M}^+(\mathbf{A})$ such that

$$(3.14) \quad S_{D_i}^{\sigma^i} = A_i \quad \text{and} \quad \langle g_i, \sigma^i \rangle > a_i \quad \text{for all } i = 1, 2;$$

these σ will serve as *constraints* for $\nu \in \mathfrak{M}^+(\mathbf{A})$. Given $\sigma \in \mathfrak{C}(\mathbf{A})$, write

$$\mathfrak{M}^\sigma(\mathbf{A}) := \{\nu \in \mathfrak{M}^+(\mathbf{A}) : \nu^i \leq \sigma^i \quad \text{for all } i = 1, 2\},$$

where $\nu^i \leq \sigma^i$ means that $\sigma^i - \nu^i$ is a nonnegative scalar measure, and

$$\mathfrak{M}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \mathfrak{M}^\sigma(\mathbf{A}) \cap \mathfrak{M}^+(\mathbf{A}, \mathbf{a}, \mathbf{g}),$$

$$\mathcal{E}_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \mathfrak{M}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g}) \cap \mathcal{E}_{\alpha, \mathbf{f}}^+(\mathbf{A}).$$

Since $\mathcal{E}_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g}) \subset \mathcal{E}_{\alpha, \mathbf{f}}^+(\mathbf{A}, \mathbf{a}, \mathbf{g})$, we get

$$-\infty < G_{\alpha, \mathbf{f}}(\mathbf{A}, \mathbf{a}, \mathbf{g}) \leq G_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g}) := \inf_{\nu \in \mathcal{E}_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g})} G_{\alpha, \mathbf{f}}(\nu) \leq \infty.$$

If the class $\mathcal{E}_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is nonempty or, equivalently, if

$$(3.15) \quad G_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g}) < \infty,$$

then the following constrained \mathbf{f} -weighted minimum α -Riesz energy problem, also known as the *constrained Gauss variational problem*, makes sense.

Problem 3.9. Given $\sigma \in \mathfrak{C}(\mathbf{A})$, does there exist $\lambda_{\mathbf{A}}^\sigma \in \mathcal{E}_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g})$ with

$$G_{\alpha, \mathbf{f}}(\lambda_{\mathbf{A}}^\sigma) = G_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g})?$$

Remark 3.10. Assume for a moment that (3.12) holds. It has been shown by [16, Theorem 6.2] that if, in addition, $g_i|_{A_i}$, $i = 1, 2$, are bounded from above and conditions (3.13) and (3.15) are satisfied, then, in both Cases I and II, Problem 3.9 is (uniquely) solvable. But this does not remain true if requirement (3.12) is dropped.

Remark 3.11. If $0 < \alpha \leq 2 < n$, $a_1 = a_2$, $\mathbf{g} = \mathbf{1}$, A_2 is not α -thin at $\omega_{\mathbb{R}^n}$, $f_2 = 0$ and $\sigma^2 = \infty$ (i.e., no external field and no constraint act on the measures concentrated in A_2), then sufficient and/or necessary conditions for the solvability of Problem 3.9 have been established in [7]. Crucial to the arguments exploited in [7] is that, in this special case, Problem 3.9 can be reduced to the problem of minimizing the f_1 -weighted $g_{D_1}^\alpha$ -Green energy over the class $\mathcal{E}_{g_{D_1}^\alpha}^+(A_1; D_1)$. However, under the assumptions of the present study, such an observation is no longer valid.

Remark 3.12. If $a_1 = a_2$, $\mathbf{g} = \mathbf{1}$, $\mathbf{f} = \mathbf{0}$ and A_i , $i = 1, 2$, are bounded, then the constrained minimum logarithmic energy problem for a condenser with touching plates in \mathbb{C} has been investigated by Beckermann and Gryson (see [1, Theorem 2.2]). Our paper is related to the α -Riesz kernels, $0 < \alpha < n$, in \mathbb{R}^n , $n \geq 2$, and the results obtained and the approaches developed are rather different from those in [1].

4. SUFFICIENT CONDITIONS FOR THE SOLVABILITY OF PROBLEM 3.9

Denote by \overline{B} the closure of $B \subset \mathbb{R}^n$ in $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\omega_{\mathbb{R}^n}\}$, the one-point compactification of \mathbb{R}^n .

Theorem 4.1. *Let \mathbf{A} , \mathbf{f} , \mathbf{g} and $\sigma \in \mathfrak{C}(\mathbf{A})$ possess the following four properties:*

- (a') $\overline{A_1} \cap \overline{A_2}$ consists of at most one point, i.e., $\overline{A_1} \cap \overline{A_2} = \emptyset \vee \{x_0\}$ where $x_0 \in \overline{\mathbb{R}^n}$;
- (b') $f_i(x) < \infty$ n.e. in A_i , $i = 1, 2$;
- (c') $E_\alpha(\sigma^i|_{K_i}) < \infty$ for every compact $K_i \subset A_i$, $i = 1, 2$;
- (d') $\langle g_i, \sigma^i \rangle < \infty$, $i = 1, 2$.

Then, in both Cases I and II, Problem 3.9 is uniquely solvable for every vector \mathbf{a} .

The proof of Theorem 4.1 is given in Section 6; it is based on Theorem 5.1, which provides a strong completeness result for metric subspaces of $\check{\mathcal{E}}_\alpha^+(\mathbf{A})$.

Example 4.2. Let $\mathbf{A} = (A_1, A_2)$ be as in Example 3.2. Having fixed $\alpha \in (0, 3)$, assume that $\mathbf{g} = \mathbf{1}$ and either Case II holds or $f_i(x) < \infty$ n.e. in A_i , $i = 1, 2$. For any $\mathbf{a} = (a_i)_{i=1,2}$ define $\sigma^i := c_i m_3|_{A_i}$, where $c_i \in (a_i, \infty)$ is chosen arbitrarily and m_3 denotes the 3-dimensional Lebesgue measure on \mathbb{R}^3 . Then, by Theorem 4.1, Problem 3.9 admits a solution; hence, no short-circuit between A_1 and A_2 occurs, though these conductors touch each other at the point ξ_0 (see Example 3.2).

5. STRONG COMPLETENESS THEOREM FOR METRIC SUBSPACES OF $\check{\mathcal{E}}_\alpha^+(\mathbf{A})$

Let $\mathfrak{M}^+(\mathbf{A}, \leq \mathbf{a}, \mathbf{g})$ consist of all $\nu \in \mathfrak{M}^+(\mathbf{A})$ such that $\langle g_i, \nu^i \rangle \leq a_i$ for all $i = 1, 2$. In view of (3.9),

$$(5.1) \quad \nu^i(A_i) \leq a_i g_{i,\inf}^{-1} < \infty \quad \text{for all } \nu \in \mathfrak{M}^+(\mathbf{A}, \leq \mathbf{a}, \mathbf{g}).$$

Hence, $\mathcal{E}_\alpha^+(\mathbf{A}, \leq \mathbf{a}, \mathbf{g}) := \mathcal{E}_\alpha^+(\mathbf{A}) \cap \mathfrak{M}^+(\mathbf{A}, \leq \mathbf{a}, \mathbf{g})$ can be thought of as a metric subspace of $\check{\mathcal{E}}_\alpha^+(\mathbf{A})$; its topology will likewise be called *strong*.

Theorem 5.1. *Suppose that a generalized condenser \mathbf{A} satisfies condition (a') of Theorem 4.1. Then the metric space $\mathcal{E}_\alpha^+(\mathbf{A}, \leq \mathbf{a}, \mathbf{g})$ is strongly complete and the strong topology on this space is finer than the induced \mathbf{A} -vague topology.*

Remark 5.2. In view of the fact that the metric space $\mathcal{E}_\alpha^+(\mathbf{A}, \leq \mathbf{a}, \mathbf{g})$ is isometric to its R -image, Theorem 5.1 has singled out a strongly complete topological subspace of the pre-Hilbert space $\mathcal{E}_\alpha(\mathbb{R}^n)$, whose elements are signed Radon measures. This is of independent interest since, according to a well-known counterexample by Cartan, the whole pre-Hilbert space $\mathcal{E}_\alpha(\mathbb{R}^n)$ is, in general, strongly incomplete.

5.1. Auxiliary results. Based on the definition of the \mathbf{A} -vague topology (see Definition 3.3), we call a set $\mathfrak{F} \subset \mathfrak{M}^+(\mathbf{A})$ *\mathbf{A} -vaguely bounded* if, for every $i = 1, 2$ and every $\varphi \in C_0(D_i)$,

$$\sup_{\nu \in \mathfrak{F}} |\nu^i(\varphi)| < \infty.$$

Lemma 5.3. *If $\mathfrak{F} \subset \mathfrak{M}^+(\mathbf{A})$ is \mathbf{A} -vaguely bounded, then it is \mathbf{A} -vaguely relatively compact.*

Proof. Since by [3, Chapter III, Section 2, Proposition 9] any vaguely bounded part of $\mathfrak{M}^+(D_i)$ is vaguely relatively compact, the lemma follows from Tychonoff's theorem on the product of compact spaces (see, e.g., [11, Chapter 5, Theorem 13]). \square

Lemma 5.4. *$\mathfrak{M}^+(\mathbf{A}, \leq \mathbf{a}, \mathbf{g})$ is \mathbf{A} -vaguely bounded and \mathbf{A} -vaguely closed; hence, it is \mathbf{A} -vaguely compact.*

Proof. Indeed, it is obvious from (5.1) that $\mathfrak{M}^+(\mathbf{A}, \leq \mathbf{a}, \mathbf{g})$ is \mathbf{A} -vaguely bounded. Fix an arbitrary $\{\nu_k\}_{k \in \mathbb{N}} \subset \mathfrak{M}^+(\mathbf{A}, \leq \mathbf{a}, \mathbf{g})$; then, by Lemma 5.3, it has an \mathbf{A} -vague cluster point ν_0 . In fact, $\nu_0 \in \mathfrak{M}^+(\mathbf{A})$, for $\mathfrak{M}^+(\mathbf{A})$ is \mathbf{A} -vaguely closed. Choose a subsequence $\{\nu_{k_m}\}_{m \in \mathbb{N}}$ of $\{\nu_k\}_{k \in \mathbb{N}}$ that converges \mathbf{A} -vaguely to ν_0 . As g_i is positive and continuous, we get

$$\langle g_i, \nu_0^i \rangle \leq \liminf_{m \rightarrow \infty} \langle g_i, \nu_{k_m}^i \rangle \leq a_i \quad \text{for all } i = 1, 2,$$

and the lemma follows. \square

Lemma 5.5. *Assume that \mathbf{A} is a standard condenser; i.e., $\overline{A_1} \cap \overline{A_2} = \emptyset \vee \{\omega_{\mathbb{R}^n}\}$. Then the metric space $\mathcal{E}_\alpha^+(\mathbf{A})$ ($= \check{\mathcal{E}}_\alpha^+(\mathbf{A})$) is strongly complete. In more detail, any strong Cauchy sequence $\{\nu_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_\alpha^+(\mathbf{A})$ converges both strongly and \mathbf{A} -vaguely to some $\nu_0 \in \mathcal{E}_\alpha^+(\mathbf{A})$, and this limit is unique.*

Proof. It is clear from (3.2) that, for a standard \mathbf{A} ,

$$\mathcal{E}_\alpha^+(\mathbf{A}) = \check{\mathcal{E}}_\alpha^+(\mathbf{A}).$$

Since $\check{\mathcal{E}}_\alpha^+(\mathbf{A})$ and $R(\check{\mathcal{E}}_\alpha^+(\mathbf{A}))$, the latter being treated as a metric subspace of the pre-Hilbert space $\mathcal{E}_\alpha(\mathbb{R}^n)$, are isometric to each other by (3.5), the lemma follows from [15] (see Theorem 1 and Corollary 1 therein). \square

5.2. Proof of Theorem 5.1. Fix a strong Cauchy sequence $\{\nu_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_\alpha^+(\mathbf{A}, \leq \mathbf{a}, \mathbf{g})$. According to Lemma 5.4, it has an \mathbf{A} -vague cluster point $\nu_0 \in \mathfrak{M}^+(\mathbf{A}, \leq \mathbf{a}, \mathbf{g})$. Let $\{\nu_{k_m}\}_{m \in \mathbb{N}}$ be a (strong Cauchy) subsequence of $\{\nu_k\}_{k \in \mathbb{N}}$ that converges \mathbf{A} -vaguely to ν_0 , i.e.

$$(5.2) \quad \nu_{k_m}^i \rightarrow \nu_0^i \quad \text{vaguely in } \mathfrak{M}(D_i), \quad i = 1, 2.$$

We proceed by showing that $E_\alpha(\nu_0)$ is finite, so that

$$(5.3) \quad \nu_0 \in \mathcal{E}_\alpha^+(\mathbf{A}, \leq \mathbf{a}, \mathbf{g}) \quad (\subset \check{\mathcal{E}}_\alpha^+(\mathbf{A})),$$

and, moreover, $\nu_{k_m} \rightarrow \nu_0$ strongly as $m \rightarrow \infty$, i.e.

$$(5.4) \quad \lim_{m \rightarrow \infty} \|\nu_{k_m} - \nu_0\|_{\check{\mathcal{E}}_\alpha^+(\mathbf{A})} = 0.$$

To establish these assertions, it is enough to analyze the case

$$(5.5) \quad \overline{A_1} \cap \overline{A_2} = \{x_0\} \quad \text{where } x_0 \in \mathbb{R}^n,$$

since otherwise they are obtained directly from Lemma 5.5.

Consider the inversion I with respect to the $(n-1)$ -dimensional unit sphere centered at x_0 ; namely, each point $x \neq x_0$ is mapped to the point x^* on the ray through x which issues from x_0 , determined uniquely by

$$|x - x_0| \cdot |x^* - x_0| = 1.$$

This is a one-to-one, bicontinuous mapping of $\mathbb{R}^n \setminus \{x_0\}$ onto itself; furthermore,

$$(5.6) \quad |x^* - y^*| = \frac{|x - y|}{|x_0 - x||x_0 - y|}.$$

Extend it to a one-to-one, bicontinuous map of $\overline{\mathbb{R}^n}$ onto itself by setting $I(x_0) = \omega_{\mathbb{R}^n}$.

To each signed scalar measure $\nu \in \mathfrak{M}(\mathbb{R}^n)$ with $\nu(\{x_0\}) = 0$ there corresponds the Kelvin transform $\nu^* \in \mathfrak{M}(\mathbb{R}^n)$ by means of the formula

$$d\nu^*(x^*) = |x - x_0|^{\alpha-n} d\nu(x), \quad x^* \in \mathbb{R}^n$$

(see [14] or [12, Chapter IV, Section 5, n° 19]). Then, in view of (5.6),

$$U_\alpha^{\nu^*}(x^*) = |x - x_0|^{n-\alpha} U_\alpha^\nu(x), \quad x^* \in \mathbb{R}^n,$$

and therefore

$$(5.7) \quad E_\alpha(\nu^*) = E_\alpha(\nu).$$

It is clear that the Kelvin transformation is additive and it is an involution, i.e.

$$(5.8) \quad (\nu_1 + \nu_2)^* = \nu_1^* + \nu_2^*,$$

$$(5.9) \quad (\nu^*)^* = \nu.$$

Write $A_i^* := I(\overline{A_i}) \cap \mathbb{R}^n$, $i = 1, 2$; then $\mathbf{A}^* = (A_1^*, A_2^*)$ forms a standard condenser in \mathbb{R}^n , which is obvious from (5.5) and the above-mentioned properties of I .

Applying the Kelvin transformation to each of the components of any given $\nu = (\nu^i)_{i=1,2} \in \mathfrak{M}^+(\mathbf{A})$, we get $\nu^* := ((\nu^i)^*)_{i=1,2} \in \mathfrak{M}^+(\mathbf{A}^*)$; and the other way around. Based on Lemma 3.4 and relations (3.5) and (5.7)–(5.9), we also see that the α -Riesz energy of $\nu \in \mathfrak{M}^+(\mathbf{A})$ is well defined if and only if so is that of ν^* , and then they coincide; and, furthermore,

$$(5.10) \quad \|\nu_1^* - \nu_2^*\|_{\mathcal{E}_\alpha^+(\mathbf{A}^*)} = \|\nu_1 - \nu_2\|_{\check{\mathcal{E}}_\alpha^+(\mathbf{A})} \quad \text{for all } \nu_1, \nu_2 \in \check{\mathcal{E}}_\alpha^+(\mathbf{A}).$$

Summarizing what has thus been observed, we conclude that the Kelvin transformation is a one-to-one, isometric mapping of $\check{\mathcal{E}}_\alpha^+(\mathbf{A})$ onto $\mathcal{E}_\alpha^+(\mathbf{A}^*)$.

Let ν_{k_m} , $m \in \mathbb{N}$, and ν_0 be as above. In view of (5.1) and (5.2), for each $i = 1, 2$ one can apply [12, Lemma 4.3] to $\nu_{k_m}^i$, $k \in \mathbb{N}$, and ν_0^i , and consequently

$$(5.11) \quad \nu_{k_m}^* \rightarrow \nu_0^* \quad \mathbf{A}\text{-vaguely as } m \rightarrow \infty.$$

But $\{\nu_{k_m}^*\}_{m \in \mathbb{N}}$ is a strong Cauchy sequence in $\mathcal{E}_\alpha^+(\mathbf{A}^*)$, which is clear from (5.10). This together with (5.11) implies, by Lemma 5.5, that $\nu_0^* \in \mathcal{E}_\alpha^+(\mathbf{A}^*)$ and

$$\lim_{m \rightarrow \infty} \|\nu_{k_m}^* - \nu_0^*\|_{\mathcal{E}_\alpha^+(\mathbf{A}^*)} = 0.$$

Repeated application of (5.10) then leads to relations (5.3) and (5.4) as claimed.

In turn, (5.4) yields $\nu_k \rightarrow \nu_0$ strongly as $k \rightarrow \infty$, for $\{\nu_k\}_{k \in \mathbb{N}}$ is strongly fundamental. It has thus been established that $\{\nu_k\}_{k \in \mathbb{N}}$ converges strongly to any of its \mathbf{A} -vague cluster points. As $\|\nu_1 - \nu_2\|_{\mathcal{E}_\alpha^+(\mathbf{A})}$ is a metric, ν_0 has to be the unique \mathbf{A} -vague cluster point of $\{\nu_k\}_{k \in \mathbb{N}}$. Since the \mathbf{A} -vague topology is Hausdorff, ν_0 is actually also the \mathbf{A} -vague limit of $\{\nu_k\}_{k \in \mathbb{N}}$ (cf. [2, Chapter I, Section 9, n° 1]). This completes the proof. \square

6. PROOF OF THEOREM 4.1

We start by observing that $\mathcal{E}_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is nonempty and, hence, (3.15) holds. Indeed, it is seen from assumptions (3.14) and (b') in consequence of [9, Lemma 1.2.2] that, for every $i = 1, 2$, there is a compact set $K_i \subset A_i$ such that $\langle g_i, \sigma^i|_{K_i} \rangle > a_i$ and $f_i(x) \leq M < \infty$ for all $x \in K_i$. Define $\theta^i := \sigma^i|_{K_i} / \langle g_i, \sigma^i|_{K_i} \rangle$. Due to assumption (c') and Lemma 3.4, we then obtain $\theta := (\theta^i)_{i=1,2} \in \mathcal{E}_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g})$ as claimed.

Therefore, the class $\mathbb{M}_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g})$ of all $\{\nu_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g})$ with

$$(6.1) \quad \lim_{k \rightarrow \infty} G_{\alpha, \mathbf{f}}(\nu_k) = G_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g})$$

is nonempty. Fix arbitrary $\{\nu_k\}_{k \in \mathbb{N}}$ and $\{\mu_m\}_{m \in \mathbb{N}}$ in $\mathbb{M}_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g})$. Taking (3.10) into account, we proceed by proving that

$$(6.2) \quad \lim_{k, m \rightarrow \infty} \|\nu_k - \mu_m\|_{\mathcal{E}_\alpha^+(\mathbf{A})} = 0.$$

Based on the convexity of $\mathcal{E}_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g})$, from (3.4) and (3.6) we get

$$4G_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g}) \leq 4G_{\alpha, \mathbf{f}}\left(\frac{\nu_k + \mu_m}{2}\right) = \|R\nu_k + R\mu_m\|_\alpha^2 + 4\langle \mathbf{f}, \nu_k + \mu_m \rangle.$$

On the other hand, applying the parallelogram identity in the pre-Hilbert space $\mathcal{E}_\alpha(\mathbb{R}^n)$ to $R\nu_k$ and $R\mu_m$ and then adding and subtracting $4\langle \mathbf{f}, \nu_k + \mu_m \rangle$, we have

$$\|R\nu_k - R\mu_m\|_\alpha^2 = -\|R\nu_k + R\mu_m\|_\alpha^2 - 4\langle \mathbf{f}, \nu_k + \mu_m \rangle + 2G_{\alpha, \mathbf{f}}(\nu_k) + 2G_{\alpha, \mathbf{f}}(\mu_m).$$

When combined with the preceding relation, this gives

$$0 \leq \|R\nu_k - R\mu_m\|_\alpha^2 \leq -4G_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g}) + 2G_{\alpha, \mathbf{f}}(\nu_k) + 2G_{\alpha, \mathbf{f}}(\mu_m).$$

On account of (3.5), (6.1) and the fact that $G_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is finite, we derive (6.2) from the very relation by letting $k, m \rightarrow \infty$.

Assuming now $\{\nu_k\}_{k \in \mathbb{N}}$ and $\{\mu_m\}_{m \in \mathbb{N}}$ in (6.2) to be equal, we see that any fixed sequence $\{\nu_k\}_{k \in \mathbb{N}} \in \mathbb{M}_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g})$ is strongly fundamental in the metric space $\mathcal{E}_\alpha^+(\mathbf{A}, \leq \mathbf{a}, \mathbf{g})$. Thus, by Theorem 5.1, there exists the unique $\nu_0 \in \mathcal{E}_\alpha^+(\mathbf{A}, \leq \mathbf{a}, \mathbf{g})$ such that

$$(6.3) \quad \nu_k \rightarrow \nu_0 \quad \mathbf{A}\text{-vaguely (as } k \rightarrow \infty),$$

$$(6.4) \quad \lim_{k \rightarrow \infty} \|\nu_k - \nu_0\|_{\mathcal{E}_\alpha^+(\mathbf{A})} = 0.$$

We assert that this ν_0 gives a solution to Problem 3.9, i.e.

$$(6.5) \quad \nu_0 \in \mathcal{E}_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g}) \quad \text{and} \quad G_{\alpha, \mathbf{f}}(\nu_0) = G_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g}).$$

Observe that

$$G_{\alpha, \mathbf{f}}(\nu_0) \leq \liminf_{k \rightarrow \infty} G_{\alpha, \mathbf{f}}(\nu_k).$$

Indeed, if Case I holds, then this inequality can be obtained directly from (6.3) and (6.4), while otherwise it follows from (6.4) with the help of (3.7). Combining it with (6.1) and (3.15), we get $G_{\alpha, \mathbf{f}}(\nu_0) \leq G_{\alpha, \mathbf{f}}^\sigma(\mathbf{A}, \mathbf{a}, \mathbf{g}) < \infty$.

As $\mathfrak{M}^\sigma(\mathbf{A})$ is \mathbf{A} -vaguely closed, we therefore conclude that relation (6.5) will have been established once for each $i = 1, 2$ we show

$$(6.6) \quad \langle g_i, \nu_0^i \rangle = a_i.$$

Consider an exhaustion of A_i by an increasing sequence of compact sets $K_\ell \subset A_i$, $\ell \in \mathbb{N}$. In view of the positivity and continuity of g_i on A_i , from (6.3) and [9, Lemma 1.2.2] we get

$$\begin{aligned} a_i &\geq \langle g_i, \nu_0^i \rangle = \lim_{\ell \rightarrow \infty} \langle g_i \chi_{K_\ell}, \nu_0^i \rangle \geq \lim_{\ell \rightarrow \infty} \limsup_{k \rightarrow \infty} \langle g_i \chi_{K_\ell}, \nu_k^i \rangle \\ &= a_i - \lim_{\ell \rightarrow \infty} \liminf_{k \rightarrow \infty} \langle g_i \chi_{A_i \setminus K_\ell}, \nu_k^i \rangle. \end{aligned}$$

Hence, to prove (6.6), it is enough to verify the relation

$$(6.7) \quad \lim_{\ell \rightarrow \infty} \liminf_{k \rightarrow \infty} \langle g_i \chi_{A_i \setminus K_\ell}, \nu_k^i \rangle = 0.$$

Since, by (d'),

$$\infty > \langle g_i, \sigma^i \rangle = \lim_{\ell \rightarrow \infty} \langle g_i \chi_{K_\ell}, \sigma^i \rangle,$$

we have

$$\lim_{\ell \rightarrow \infty} \langle g_i \chi_{A_i \setminus K_\ell}, \sigma^i \rangle = 0.$$

When combined with

$$\langle g_i \chi_{A_i \setminus K_\ell}, \nu_k^i \rangle \leq \langle g_i \chi_{A_i \setminus K_\ell}, \sigma^i \rangle \quad \text{for all } \ell, k \in \mathbb{N},$$

this implies (6.7), hence (6.6), and consequently (6.5).

It is left to establish the statement on the uniqueness. Let, on the contrary, $\hat{\nu}_0$ be an other solution of Problem 3.9. Then trivial sequences $\{\nu_0\}$ and $\{\hat{\nu}_0\}$ are both elements of

$\mathbb{M}_{\alpha, \mathbf{f}}^{\sigma}(\mathbf{A}, \mathbf{a}, \mathbf{g})$ and therefore, by (6.2), $\|\nu_0 - \widehat{\nu}_0\|_{\mathcal{E}_{\alpha}^{+}(\mathbf{A})} = 0$. As $\mathcal{E}_{\alpha}^{+}(\mathbf{A})$ is a metric space, this results in $\nu_0 = \widehat{\nu}_0$, and the proof is complete. \square

REFERENCES

- [1] B. Beckermann, A. Gryson, *Extremal rational functions on symmetric discrete sets and superlinear convergence of the ADI method*, Constr. Approx. **32** (2010), 393–428.
- [2] N. Bourbaki, *Elements of Mathematics. General Topology. Chap. 1–4*, Springer, Berlin, 1989.
- [3] N. Bourbaki, *Elements of Mathematics. Integration. Chap. 1–6*, Springer, Berlin, 2004.
- [4] H. Cartan, *Théorie du potentiel Newtonien: énergie, capacité, suites de potentiels*, Bull. Soc. Math. Fr. **73** (1945), 74–106.
- [5] J. Deny, *Les potentiels d'énergie finie*, Acta Math. **82** (1950), 107–183.
- [6] J. Deny, *Sur la définition de l'énergie en théorie du potentiel*, Ann. Inst. Fourier Grenoble **2** (1950), 83–99.
- [7] P.D. Dragnev, D.P. Hardin, E.B. Saff, N. Zorii, *Minimum Riesz energy problems for a condenser with "touching plates"*, ArXiv:1504.03805 (2015), 32 p.
- [8] R. Edwards, *Functional analysis. Theory and applications*, Holt, Rinehart and Winston, New York, 1965.
- [9] B. Fuglede, *On the theory of potentials in locally compact spaces*, Acta Math. **103** (1960), 139–215.
- [10] C.F. Gauss, *Allgemeine Lehrsätze in Beziehung auf die im verkehrten Verhältnisse des Quadrats der Entfernung wirkenden Anziehungs- und Abstoßungs-Kräfte* (1839), Werke **5** (1867), 197–244.
- [11] J.L. Kelley, *General Topology*, Princeton, New York (1957).
- [12] N.S. Landkof, *Foundations of Modern Potential Theory*, Springer, Berlin (1972).
- [13] M. Ohtsuka, *On potentials in locally compact spaces*, J. Sci. Hiroshima Univ. Ser. A-1 **25** (1961), 135–352.
- [14] M. Riesz, *Intégrales de Riemann–Liouville et potentiels*, Acta Szeged, **9** (1938), 1–42.
- [15] N. Zorii, *A noncompact variational problem in Riesz potential theory. I; II*, Ukr. Math. J. **47** (1995), 1541–1553; **48** (1996), 671–682.
- [16] N. Zorii, *Constrained energy problems with external fields for vector measures*, Math. Nachr. **285** (2012), 1144–1165.
- [17] N. Zorii, *Equilibrium problems for infinite dimensional vector potentials with external fields*, Potential Anal. **38** (2013), 397–432.
- [18] N. Zorii, *Necessary and sufficient conditions for the solvability of the Gauss variational problem for infinite dimensional vector measures*, Potential Anal. **41** (2014), 81–115.

INSTITUTE OF MATHEMATICS OF NATIONAL ACADEMY OF SCIENCES OF UKRAINE, TERESHCHENKIVSKA 3, 01601, KYIV-4, UKRAINE

E-mail address: natalia.zorii@gmail.com